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Relative FBN rings and the second layer condition

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Abstract

Various versions of the second layer condition and the relationships between them are studied. It is proved that a right noetherian ring R satisfies the right second layer condition iff every finitely generated tame right R-module is a Δ-module, iff R is right fully tame bounded, meaning that every essential right ideal E/P of a prime factor ring R/P contains a nonzero ideal of R/P whenever R/E is tame. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Since it was first introduced, the second layer condition has become one of the most important concepts for the study of non-commutative noetherian rings. Roughly speaking, it postulates that if $0 \to L \to M \to N \to 0$ is a non-split exact sequence of R-modules, such that L and N are modules over prime factor rings, then N should not be torsion if L is torsionfree. Such non-split extensions of tame modules by tame modules lead to the notion of links between prime ideals, and these links have been the object of intense study, since they constitute a major obstruction to localization.

More specifically, following Jategaonkar [6], a prime ideal P of a right noetherian ring is said to satisfy the *right second layer condition* if the second layer of the injective hull $\mathrm{E}_R(R/P)$ of the right R-module R/P is tame, that is, if every prime submodule of $\mathrm{E}_R(R/P)/\ell_{\mathrm{E}_R(R/P)}(P)$ is torsionfree. In [6], another condition was introduced, called the *right strong second layer condition*, and in the literature at least three variations of one or the other of these conditions can be found, with various names.

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One of the objectives of this note is to present different formulations of these second layer conditions and to clarify as much as possible how they are related to each other. Some of this is well-known for (two-sided) noetherian rings, but we restrict ourselves to right noetherian rings, where the picture, up till now, has perhaps been less clear.

Certain prime ideals of a right noetherian ring automatically satisfy the right second layer condition, for example, every minimal prime does, which was first observed by Boyle and Kosler [1], and follows from our Proposition 5.4. Many noetherian rings satisfy the second layer condition on both sides, that is, all their prime ideals satisfy it. This is the case for enveloping algebras of finite dimensional solvable Lie algebras, group rings of polycyclic-by-finite groups (over commutative noetherian coefficient rings), as well as noetherian PI-rings. However, noetherian rings exist that do not satisfy this condition, enveloping algebras of semisimple Lie algebras being one such instance. An important class of right noetherian rings that do satisfy the right second layer condition are the right fully bounded ones. Such right FBN rings have been characterized by Cauchon [2] as those right noetherian rings R for which every finitely generated right R-module M is a Δ-module, that is, R satisfies the descending chain condition for annihilators of subsets of M. Now, Shapiro [11] has proved that a prime ideal P of a right noetherian ring R satisfies a variant of the right second layer condition if and only if every finitely generated P-tame right R-module M is finitely annihilated, that is, the descending chain condition holds for annihilators of submodules of M. Therefore, the question arises, whether the formal resemblance between right FBN rings and right noetherian rings satisfying the right second layer condition can be made more precise. To show that this is indeed possible constitutes the other main objective of this paper. Thus, a right noetherian ring R is shown to satisfy the right second layer condition if and only if every finitely generated tame right R-module is a Δ-module, if and only if every prime homomorphic image $\overline{R}$ of R is right tame bounded, that is, any essential right ideal $\overline{E}$ of $\overline{R}$ for which $\overline{R}/\overline{E}$ is tame contains a non-zero two-sided ideal (Theorem 6.2).

We hope that this characterization of the second layer condition will make it easier to verify the condition in concrete examples. A major drawback of the original definition is that it is not easy to check, since generally one does not have a good grip on the structure of the injective module $E_R(R/P)$. This presents a formidable obstacle when one tries to verify the second layer condition, even for a "nice" extension of a ring that satisfies it. Thus, for example, the case of a skew polynomial ring $R[x;\sigma]$ where R is a noetherian ring with second layer condition, still remains unsettled in general.

2. Definitions and notations

All rings considered are associative with unit element 1, modules are unitary. For standard terminology the reader is referred to [3, 10].

Let M be a right R-module, and let X and Y be subsets of M and R, respectively. Then

$$\ell_M(Y) = \text{annihilator of } Y \text{ in } M = \{m \in M \mid mY = 0\},$$

$$r_R(X) = \text{annihilator of } X \text{ in } R = \{r \in R \mid Xr = 0\}.$$

The subscripts will be deleted if there is no danger of ambiguity. A prime ideal P of R is *associated* with the right R-module M if there exists a submodule $0 \neq N \subseteq M$ such that $P = r(N')$ for all submodules $0 \neq N' \subseteq N$.

mod-R = category of right R-modules
Ass(M) = set of associated primes of the R-module M
Spec(R) = set of all prime ideals of R
$\text{Spec}_\tau(R) = \{P \in \text{Spec}(R) \mid R/P \text{ is } \tau\text{-torsionfree}\}$, τ a hereditary torsion theory on mod-R
$\text{annSpec}(M) = \{P \in \text{Spec}(R) \mid P = r(N) \text{ for some } N \subseteq M\}$
$N \subseteq_{\text{ess}} M$ = N is an essential submodule of M
$\text{E}_R(M) = \text{E}(M)$ = injective envelope of the right R-module M
E_P = injective indecomposable direct summand of $\text{E}_R(R/P)$
$\rho(M)$ = reduced rank of the module M
$\text{dev}(\mathcal{S})$ = deviation of the partially ordered set $\mathcal{S}$
$|M| = |M|_R$ = Krull dimension of the right R-module M
$\kappa_\tau(M)$ = relative Krull dimension of M with respect to the hereditary torsion theory τ
Cl.K.dim(R) = classical Krull dimension of the ring R

If I is an ideal of the ring R, then

$$\mathscr{C}'(I) = \{c \in R \mid cx \in I \text{ implies that } x \in I\},$$

$$'\mathscr{C}(I) = \{c \in R \mid xc \in I \text{ implies that } x \in I\},$$

$$\mathscr{C}(I) = {}'\mathscr{C}(I) \cap \mathscr{C}'(I).$$

A right R-module M is called *P-primary* if Ass(M) = P; it is called *P-prime* if Ass(M) = P = $r(M)$. A uniform P-primary right R-module U is *P-tame*, or simply *tame*, if the P-prime submodule $\ell_M(P)$ is torsionfree as a right R/P-module, that is, no non-zero element of $\ell_M(P)$ is annihilated by an element of $\mathscr{C}(P)$; it is called *P-wild*, or simply *wild*, if $\ell_M(P)$ is $\mathscr{C}(P)$-torsion. A right R-module is *tame* (*wild*) if all its uniform submodules are tame (wild). A right R-module M is *$\mathscr{X}$-tame* for a set $\mathscr{X}$ of prime ideals, if it is tame and Ass(M) $\subseteq \mathscr{X}$. Note that submodules, essential extensions and direct sums of tame modules are tame. It is also easy to see that extensions of tame modules by tame modules are tame.

A right R-module M is called *finitely annihilated* if there exist elements $m_1, m_2, \ldots, m_n \in M$ such that $r(M) = r(m_1, \ldots, m_n) = \bigcap_{i=1}^n r(m_i)$; it is a *$\Delta$-module* if R satisfies the descending chain condition for right annihilators of subsets of M. Note that a Δ-module is tame.

An *affiliated series* of a right R-module M is a sequence of submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M$, together with a set of prime ideals $\{P_1, \ldots, P_n\}$ called *affiliated primes* such that each P_i is maximal in $\mathrm{Ass}(M/M_{i-1})$ and $M_i/M_{i-1} = \ell_{M/M_{i-1}}(P_i)$.

Let R be a ring, and let $Q, P \in \mathrm{Spec}(R)$. Then Q is *linked to* P (*via* A), denoted by $Q \rightsquigarrow P$, if A is an ideal with $QP \subseteq A \subset Q \cap P$, such that $Q \cap P/A$ is torsionfree as a right R/P-module and fully faithful (that is, has no non-zero unfaithful submodules) as a left R/Q-module.

A subset $\mathscr{X}$ of $\mathrm{Spec}(R)$ is said to be *right link closed* if $P \in \mathscr{X}$ and $Q \rightsquigarrow P$, imply that $Q \in \mathscr{X}$.

3. Strong second layer condition

In [5, pp. 23 and 24], Jategaonkar introduced the condition $(\substack{*\\*})_r$, later calling it the *right strong second layer condition* in his book [6, p. 220].

Definition. A prime ideal P of a right noetherian ring R satisfies the *right strong second layer condition* if for every prime ideal $Q \subset P$ every finitely generated P/Q-primary right R/Q-module is unfaithful over R/Q. A set $\mathscr{X}$ of prime ideals of R satisfies the *right strong second layer condition* if every $P \in \mathscr{X}$ does so. The ring R satisfies the *right strong second layer condition* if $\mathrm{Spec}(R)$ satisfies this condition.

Some of the following characterizations of the right strong second layer condition are well-known, at least when R is (left and right) noetherian. For example, the equivalence of (i) and (iv) can be found in [3, Corollary 11.5, Exercise 11K] for the case when $\mathscr{X} = \mathrm{Spec}(R)$.

Proposition 3.1. *The following statements are equivalent for a set $\mathscr{X}$ of prime ideals of a right noetherian ring R.*

(i) *$\mathscr{X}$ satisfies the right strong second layer condition.*

(ii) *If $P \in \mathscr{X}$ and M is a finitely generated P-primary right R-module, then M^I is P-primary for any set $I \neq \emptyset$.*

(iii) $\mathrm{Ass}(M) = \mathrm{Ass}(M^I)$ *for any finitely generated right R-module M with* $\mathrm{Ass}(M) \subseteq \mathscr{X}$ *and any set $I \neq \emptyset$.*

(iv) $\mathrm{annSpec}(M) = \mathrm{Ass}(M)$ *for any finitely generated right R-module M with* $\mathrm{Ass}(M) \subseteq \mathscr{X}$.

(v) *Given any $P \in \mathscr{X}$, there does not exist a finitely generated uniform P-primary right R-module M such that $r(M/\ell_M(P)) = Q$ is a prime ideal, $Q \subset P$, and $MQ = 0$.*

Proof. (i) $\Rightarrow$ (ii): Let $P \in \mathscr{X}$, let M be a finitely generated P-primary right R-module, and let $Q \in \mathrm{Ass}(M^I)$. Choose $n = (n_i)_{i \in I} \in M^I$, such that $Q = r(nR) = \bigcap_{i \in I} r(n_i R) = r(\sum_{i \in I} n_i R)$. Since M is noetherian, $N = \sum_{i \in I} n_i R \subseteq M$ is a finitely generated

P-primary module. As P is assumed to satisfy the right strong second layer condition, $Q = P$ follows.

(ii) $\Rightarrow$ (iii): Let M be a finitely generated right R-module with $\mathrm{Ass}(M) = \{P_1, \ldots, P_n\} \subseteq \mathscr{X}$. Choose submodules $N_j \subseteq M$, such that $\bigcap_{j=1}^n N_j = 0$ and each M/N_j is P_j-primary. Then $M^I \hookrightarrow \bigoplus_{j=1}^n (M/N_j)^I$, so if $Q \in \mathrm{Ass}(M^I)$, then $Q \in \mathrm{Ass}((M/N_j)^I)$ for some j. By hypothesis, $(M/N_j)^I$ is P_j-primary, so $Q = P_j \in \mathrm{Ass}(M)$. Since $\mathrm{Ass}(M) \subseteq \mathrm{Ass}(M^I)$ in any case, the claim follows.

(iii) $\Rightarrow$ (iv): Let $Q \in \mathrm{annSpec}(M)$, $Q = r(N)$ for $N \subseteq M$. As $Q = \bigcap_{n \in N} r(n)$, $R/Q \hookrightarrow N^N \subseteq M^N$, whence $Q \in \mathrm{Ass}(M^N) = \mathrm{Ass}(M)$.

(iv) $\Rightarrow$ (v): Let $P \in \mathscr{X}$. Suppose there exists a finitely generated uniform right R-module M with $\mathrm{Ass}(M) = P$, $r(M/\ell_M(P)) = Q$ a prime ideal, $Q \subset P$, and $MQ = 0$. Then $Q = r(M) \in \mathrm{annSpec}(M) = \mathrm{Ass}(M) = P$, a contradiction.

(v) $\Rightarrow$ (i): Let $P \in \mathscr{X}$, and let M be a finitely generated P/Q-primary right R/Q-module for a prime ideal $Q \subset P$. Assume that M is faithful over R/Q, that is, $Q = r(M_R)$. Since M is noetherian, there exist submodules $N_i, i = 1, \ldots, n$, such that $\bigcap_{i=1}^n N_i = 0$ is an irredundant intersection and each M/N_i is uniform. Then $M \hookrightarrow_{\mathrm{ess}} \bigoplus_{i=1}^n M/N_i$. Since $\mathrm{Ass}(M/N_i) = P$ for all i and since $r(M/N_i) = Q$ for some i, we may assume that M is uniform. Now, $r(M/\ell_M(P))P \subseteq r(M) = Q$, and since $P \not\subseteq Q$, this implies $r(M/\ell_M(P)) \subseteq Q$. Since $Q = r(M) \subseteq r(M/\ell_M(P))$, it follows that $r(M/\ell_M(P)) = Q$, contradicting (v). □

Part (v) of the preceding result clarifies the connection with the strong second layer condition and another version of it that appears in [3, Theorem 11.1] and arises from the following formulation of Jategaonkar's Main Lemma [6, 6.1.3] for noetherian rings.

Lemma 3.2. *Let R be a noetherian ring, and let M be a right R-module with an affiliated series $0 \subset U \subset M$ and corresponding affiliated prime ideals P and Q, such that $U \subseteq_{\mathrm{ess}} M$. Let M' be a submodule of M, properly containing U, such that the ideal $A = r(M')$ is maximal among annihilators of submodules of M properly containing U. Then exactly one of the following two alternatives occurs:*

(i) *$Q \subset P$ and $M'Q = 0$. In this case, M' and M'/U are faithful torsion R/Q-modules.*

(ii) *$Q \rightsquigarrow P$ and $Q \cap P/A$ is a linking bimodule between Q and P. In this case, if U is torsionfree as a right R/P-module then M'/U is torsionfree as a right R/Q-module.*

In [3, p.183] a prime ideal P of a noetherian ring R is said to satisfy the *strong second layer condition* if, given the hypotheses of the preceding lemma, the conclusion (i) never occurs. However, a distinction has to be made between this concept and the previous one defined above, even for noetherian rings, since [3, Exercise 11M] shows that a prime ideal P may satisfy the latter right strong second layer condition, but does not satisfy the earlier one. Thus we shall introduce the name *right strong affiliated second layer condition* for the latter. It can be characterized as follows.

Proposition 3.3 (Goodearl and Warfield [3, Proposition 11.3]). *The following statements are equivalent for a prime ideal P of a noetherian ring.*

(i) *P satisfies the right strong affiliated second layer condition.*

(ii) *There does not exist a finitely generated uniform right R-module M with an affiliated series $0 \subset U \subset M$ and corresponding affiliated prime ideals P and Q such that M/U is uniform, $Q \subset P$, and $MQ = 0$.*

Statement (ii) of the preceding proposition will be used to define the right strong affiliated second layer condition for right noetherian rings.

Definition. A prime ideal P of a right noetherian ring R satisfies the *right strong affiliated second layer condition* if there does not exist a finitely generated uniform right R-module M with an affiliated series $0 \subset U \subset M$ and corresponding affiliated prime ideals P and Q such that M/U is uniform, $Q \subset P$, and $MQ = 0$. The ring R satisfies the *right strong affiliated second layer condition* if every prime ideal of R satisfies this condition.

It is clear from Proposition 3.1(v) that a prime ideal of a right noetherian ring that satisfies the right strong second layer condition also satisfies the right strong affiliated second layer condition. As has been pointed out above, the converse does not necessarily hold for a single prime. However, we have the following.

Proposition 3.4. *A right noetherian ring satisfies the right strong second layer condition if and only if it satisfies the right strong affiliated second layer condition.*

Proof. Assume that R satisfies the right strong affiliated second layer condition. Let P and Q be prime ideals, $Q \subset P$, and assume there exists a finitely generated uniform P-primary right R-module M with $r(M/\ell_M(P)) = Q$ and $MQ = 0$. Passing to the appropriate factor module, we may assume that there exists no proper uniform homomorphic image M' of M with $r(M') = Q$ that is P'-primary for some prime ideal $P' \supset Q$. Set $L = \ell_M(P)$, so $r(M/L) = Q$. Let $L = \bigcap_{i=1}^{n} L_i$ be an irredundant intersection, such that each M/L_i is uniform. Then $M/L \hookrightarrow_{\text{ess}} \bigoplus_{i=1}^{n} M/L_i$. Clearly, $r(M/L_i) = Q$ for some i. Let $\text{Ass}(M/L_i) = P_i$, so $P_i \supseteq Q$, and $P_i = Q$ follows since M/L_i is a proper homomorphic image of M. Thus $Q = P_i = \text{Ass}(M/L_i) \subseteq \text{Ass}(M/L)$. Now let X/L be a uniform Q-prime submodule of M/L. Then $O \subset L \subset X$ is an affiliated series of X with corresponding affiliated primes P and Q. Since X is uniform and $XQ \subseteq MQ = 0$, this is impossible. □

4. Restricted strong second layer condition

The definition of the right strong second layer condition for a prime ideal P postulates that for any prime ideal $Q \subset P$ every finitely generated P-primary R/Q-module M is unfaithful over R/Q. If one restricts the class of modules in this definition from

P-primary to P-tame, one obtains the condition $(*)_r$ of [5, p. 24]. In the following lemma we present several equivalent formulations of this condition. In its form (iv) it was labeled (†) in [3, Exercise 11L], whereas in its form (v) it was called the *right second layer condition* by Shapiro [11] (Shapiro assigns the name *right alternate second layer condition* to the condition that the second layer of $E_R(R/P)$ be tame, a condition that was first called the *right second layer condition* by Jategaonkar [6, p. 188], and will be so named in this note). We have chosen the name *right restricted strong second layer condition* for the concept under investigation in this section, hoping that this will not add to the confusion that seems to surround the choice of names for these various conditions.

Definition. A prime ideal P of a right noetherian ring R satisfies the *right restricted strong second layer condition* if for every prime ideal $Q \subset P$, every finitely generated P-tame right R/Q-module M is unfaithful over R/Q. A set $\mathscr{X}$ of prime ideals satisfies the *right restricted strong second layer condition* if every $P \in \mathscr{X}$ satisfies this condition. The ring R satisfies the *right restricted strong second layer condition* if Spec(R) does so.

Proposition 4.1. *The following conditions are equivalent for a prime ideal P of a right noetherian ring R.*

(i) *P satisfies the right restricted strong second layer condition.*

(ii) *If M is a finitely generated P-tame right R-module, then $\mathrm{Ass}(M^I) = P$ for any set $I \neq \emptyset$.*

(iii) *If M is a finitely generated P-tame right R-module, then M^I is P-tame for any set $I \neq \emptyset$.*

(iv) *Whenever M is a finitely generated submodule of $E_R(R/P)$ containing R/P such that $r_R(M)$ is a prime ideal, then $r_R(M) = P$.*

(v) *Whenever M is a finitely generated submodule of E_P, then $r(M)$ is not equal to a prime ideal strictly contained in P.*

(vi) *There does not exist a finitely generated uniform P-tame right R-module M such that $r(M/\ell_M(P)) = Q$ is a prime ideal, $Q \subset P$, and $MQ = 0$.*

Proof. (i) ⇒ (ii): This is proved just like the implication (i) ⇒ (ii) of Proposition 3.1.

(ii) ⇒ (iii): Suppose that there exists a P-prime submodule xR of M^I that is $\mathscr{C}(P)$-torsion, so $r(x)/P \subseteq_{\text{ess}} R/P$. If $x = (m_i)_{i \in I}$, then $r(x) = \bigcap_{i \in I} r(m_i)$, so each $r(m_i)/P$ is essential in R/P, and $m_i R$ is a P-prime, $\mathscr{C}(P)$-torsion submodule of M. Since M is assumed to be P-tame, this is impossible.

(iii) ⇒ (iv): Let $M \subseteq E_R(R/P)$, and assume that $r(M) = Q$ is a prime ideal, $Q \subseteq P$. Note that M is P-tame. As $R/r(M) = R/\bigcap_{m \in M} r(m) \hookrightarrow M^M$, and as M^M is P-tame by hypothesis, $R/Q = R/r(M)$ is P-tame, so $Q = P$.

(iv) ⇒ (v): Let M be a finitely generated submodule of E_P, and assume that $r(M) = Q$ is a prime ideal, $Q \subseteq P$. Since $E_P \hookrightarrow E_R(R/P)$, the module $N = M + (R/P)$ is a finitely generated submodule of $E_R(R/P)$ containing R/P, so $P = r_R(N) = r_R(M) \cap r_R(R/P) = Q \cap P = Q$.

(v) $\Rightarrow$ (vi): Assume that for some prime ideal $Q \subset P$ there exists a finitely generated uniform P-tame right R-module M such that $r(M/\ell_M(P)) = Q$ and $MQ = 0$. Then $r(M) = Q$, and since M is P-tame uniform, M embeds in E_P. By (v) this is impossible.

(vi) $\Rightarrow$ (i): Let M be a finitely generated P-tame right R-module, and assume that $r(M) = Q$ for some prime ideal $Q \subset P$. Since M is noetherian, there exist finitely many submodules $N_i \subseteq M$ such that each M/N_i is uniform and $\bigcap_{i=1}^k N_i = 0$ is an irredundant intersection. Then M embeds as an essential submodule in $\bigoplus_{i=1}^k M/N_i$, so each M/N_i is P-tame. Now $Q = r(M) = \bigcap_{i=1}^k r(M/N_i)$, so $Q = r(M/N_i)$ for some i. Thus, it may be assumed that M is uniform. Clearly $Q = r(M) \subseteq r(M/\ell_M(P))$, and since $r(M/\ell_M(P))P \subseteq Q$, yet $Q \subset P$, it follows that $r(M/\ell_M(P)) \subseteq Q$ also, showing that $r(M/\ell_M(P)) = Q$, which contradicts (vi). □

In some sense, the following result is a continuation of the above list of characterizations of the right restricted strong second layer condition. However, while the preceding descriptions of this condition are perhaps somewhat superficial, the theorem below goes deeper and presents the right restricted strong second layer condition as a form of weak fully boundedness. For this, we introduce the following concept.

Definition. Let R be a right noetherian ring, and let $\mathscr{X}$ be a set of prime ideals. The ring R is called *right $\mathscr{X}$-tame bounded* if $r(R/E) \neq 0$ for every essential right ideal E such that R/E is $\mathscr{X}$-tame. If every prime homomorphic image of R is right $\mathscr{X}$-tame bounded, then R is called *right fully $\mathscr{X}$-tame bounded.* A right (fully) Spec(R)-tame bounded ring is called *right (fully) tame bounded.*

Theorem 4.2. *The following statements are equivalent for a set $\mathscr{X}$ of prime ideals of the right noetherian ring R.*

(i) *$\mathscr{X}$ satisfies the right restricted strong second layer condition.*

(ii) *For any $\mathscr{X}$-tame finitely generated right R-module M and any set $I \neq \emptyset$ the module M^I is* Ass(M)*-tame.*

(iii) *Any finitely generated $\mathscr{X}$-tame right R-module is a Δ-module.*

(iv) *Any finitely generated $\mathscr{X}$-tame right R-module is finitely annihilated.*

(v) *Given a hereditary torsion theory τ on* mod-R *with $\mathscr{X} \subseteq \mathrm{Spec}_\tau(R)$, then $\kappa_\tau(M) = \kappa_\tau(R/r(M))$ for every finitely generated $\mathscr{X}$-tame right R-module M.*

(vi) *R is right fully $\mathscr{X}$-tame bounded.*

(vii) *Every factor ring of R is right $\mathscr{X}$-tame bounded.*

Proof. (i) $\Rightarrow$ (ii): Let M be a finitely generated $\mathscr{X}$-tame right R-module, let $\mathrm{Ass}(M) = \{P_1, \ldots, P_n\} \subseteq \mathscr{X}$. There exist submodules $N_i \subseteq M$ such that $\mathrm{Ass}(M/N_i) = P_i$ and M embeds as an essential submodule in $\bigoplus_{i=1}^n M/N_i$. Since essential extensions and submodules of tame modules are tame, each M/N_i is P_i-tame, so $(M/N_i)^I$ is P_i-tame by Lemma 4.1(iii). Consequently, $\bigoplus_{i=1}^n (M/N_i)^I$ is Ass(M)-tame. As M^I embeds in the latter module, the claim follows.

(ii) ⇒ (iii): Let M be a finitely generated $\mathscr{X}$-tame right R-module with $\mathrm{Ass}(M) = \{P_1, \ldots, P_n\} \subseteq \mathscr{X}$. Again, choose submodules N_i, $i = 1, \ldots, n$, such that $\mathrm{Ass}(M/N_i) = P_i$ and $M \hookrightarrow_{\mathrm{ess}} \bigoplus_{i=1}^n M/N_i$. Note that each M/N_i is P_i-tame. Since finite direct sums and submodules of Δ-modules are Δ-modules, we may thus assume that M is P-tame with $P \in \mathscr{X}$. Now let $S \neq \emptyset$ be a subset of M. Since $R/r(S) \hookrightarrow M^S$, $R/r(S)$ is P-tame, whence $L/r(S) = \ell_{R/r(S)}(P) \subseteq_{\mathrm{ess}} R/r(S)$. Now let $s_1 \in S$. If $A_1 = L \cap r(s_1) = r(S)$, it follows that $r(s_1) = r(S)$, and it is done. Otherwise there exists an element $s_2 \in S$ such that $A_1 \not\subseteq r(s_2)$. Set $A_2 = L \cap r(s_1) \cap r(s_2) \subset A_1$. If $A_2 = r(S)$, it is done, otherwise continue in this fashion to obtain a strictly descending chain of right ideals $A_1 \supset A_2 \supset \cdots \supset A_i \supset A_{i+1} \supset \cdots \supseteq r(S)$, where $A_i = L \cap r(s_1, s_2, \ldots, s_i)$. Note that $A_i/A_{i+1} = A_i/A_i \cap r(s_{i+1}) \simeq s_{i+1}A_i$ and $s_{i+1}A_iP \subseteq s_{i+1}LP \subseteq s_{i+1}r(S) = 0$, so $s_{i+1}A_i \subseteq \ell_M(P)$. Since M is P-tame, it follows that $\rho_{R/P}(A_i/A_{i+1}) = \rho_{R/P}(s_{i+1}A_i) > 0$ whenever $A_i \supset A_{i+1}$. Since R is right noetherian, $\rho_{R/P}(L/r(S)) < \infty$, so by the additivity of the reduced rank, the chain of the A_i's can have at most $\rho_{R/P}(L/r(S))$ proper inclusions. Thus $A_i = r(S)$ for some i, whence $r(S) = r(s_1, s_2, \ldots, s_i)$.

(iii) ⇒ (iv): This is trivial.

(iv) ⇒ (v): By [4, Corollary 1.6], $\kappa_\tau(M) \leq \kappa_\tau(R/r(M))$ for any finitely generated right R-module M. Since M is assumed to be finitely annihilated, $R/r(M) \hookrightarrow M^n$ for some integer $n \geq 1$, so the reverse inequality follows as well.

(v) ⇒ (vi): Let P be a prime ideal and let E/P be an essential right ideal of R/P such that R/E is $\mathscr{X}$-tame. Assume that $\kappa_\tau(R/P) = -1$, that is, R/P is τ-torsion, whence R/E is τ-torsion. Let $Q \in \mathrm{Ass}(R/E)$. Then $Q \in \mathscr{X} \subseteq \mathrm{Spec}_\tau(R)$, making R/Q τ-torsionfree. Since R/E is $\mathscr{X}$-tame, some non-zero submodule N of R/E is isomorphic to a uniform right ideal of R/Q. Since N is τ-torsion, this leads to a contradiction. Thus $\kappa_\tau(R/P) > -1$. Since E/P contains a regular element of R/P by Goldie's Theorem, it follows that $\kappa_\tau(R/r(R/E)) = \kappa_\tau(R/E) < \kappa_\tau(R/P)$, hence $r(R/E) \supset P$.

(vi) ⇒ (i): Let $P \in \mathscr{X}$, let M be a finitely generated submodule of E_P with prime annihilator Q, and suppose that $Q \subset P$. Obviously, it may be assumed that M is cyclic, $M = mR$. Since $R/r(m) \simeq mR \subseteq E_P$ is P-tame, it follows from (vi) that $r(m)/Q$ is not essential in R/Q, so $\mathrm{E}_R(R/r(m)) \hookrightarrow \mathrm{E}_R(R/Q)$, so $\mathrm{Ass}(M) = Q$. Since $\mathrm{Ass}(M) = \mathrm{Ass}(E_P) = P$, this leads to a contradiction.

(iv) ⇒ (vii): It is obviously sufficient to establish this for R. Let E be an essential right ideal of R such that R/E is $\mathscr{X}$-tame. Note that the right annihilator of each coset $r + E$ is also an essential right ideal. Since $r(R/E)$ is a finite intersection of these by hypothesis, it follows that $r(R/E) \supset 0$.

Finally, the implication (vii) ⇒ (vi) is trivial. □

Note that Shapiro [11, Proposition 1.2] proved that a prime ideal P satisfies the right restricted strong second layer condition if and only if every finitely generated P-tame right R-module M is finitely annihilated. Our proof that M is even a Δ-module is a sharpening of his argument, made possible by first characterizing the condition as in part (ii) of the preceding result.

5. The second layer condition

Before we define the second layer condition, we recall Jategaonkar's original Main Lemma [6, 6.1.3].

Lemma 5.1. *Let M be a P-tame right module over a right noetherian ring R. Set $A=r(M)$, and let $L=\ell_M(P)$. Assume that for any submodule $N\subseteq M$ either $N\subseteq L$ or $r(N)=A$. Assume further that the module M/L is uniform and is annihilated by its associated prime, say Q. Then one of the following two exclusive assertions holds*:

(i) The desirable case: *$Q\rightsquigarrow P$ via A. In this case, if the elements of $\mathscr{C}(Q)$ are non-zero-divisors on the left R-module $Q\cap P/A$ then M/L is Q-tame. In particular, this holds if R is a noetherian ring.*

(ii) The undesirable case: *$A=Q\subset P$. In this case, M/L is a wild module, and both M and M/L are faithful and torsion as right modules over R/Q.*

Definition. A prime ideal P of the right noetherian ring R satisfies the *right affiliated second layer condition* if, in the setup of Lemma 5.1, the undesirable case (ii) never occurs.

This definition is motivated by the following lemma, which should be compared to the corresponding result [3, Proposition 11.3(b)] for noetherian rings.

Lemma 5.2. *The following statements are equivalent for a prime ideal of the right noetherian ring R.*

(i) *P satisfies the right affiliated second layer condition.*

(ii) *There does not exist a finitely generated P-tame uniform right R-module M with an affiliated series $0\subset L\subset M$ and corresponding affiliated prime ideals P and Q such that M/L is uniform, $Q\subset P$, and $MQ=0$.*

Proof. (i)$\Rightarrow$(ii): Suppose that there exists a finitely generated P-tame right R-module M as described in (ii). Replacing M, if needed, by a submodule $M'\not\subseteq L$ whose right annihilator is maximal among right annihilators of submodules that are not contained in L, it may be assumed that $r(N)=r(M)$ for all submodules $N\not\subseteq L$. Thus M satisfies the setup of Lemma 5.1, and the undesirable case occurs. This, however, cannot happen if P is assumed to satisfy the right affiliated second layer condition.

(ii)$\Rightarrow$(i): (see [3, Proposition 11.3]). Assume that there exists a P-tame right R-module M satisfying the hypotheses of Lemma 5.1, such that the undesirable case occurs. Note that $L=\ell_M(P)\subseteq_{\text{ess}}M$, and also note that we may replace M by mR for some $m\in M\backslash L$, so we may assume that M is finitely generated. Now $\mathrm{E}(M)=\bigoplus_{i=1}^n E_i$ for injective indecomposable right R-modules E_i (in fact, as M is P-tame, $E_i\simeq E_P$ for all i). For each i, set $L_i=\ell_{E_i}(P)$, $K_i=\ell_{E_i}(Q)$, so that $L'=\bigoplus_{i=1}^n L_i=\ell_{\mathrm{E}(M)}(P)$ and $K'=\bigoplus_{i=1}^n K_i=\ell_{\mathrm{E}(M)}(Q)$. Set $M'=M+L'$, and observe that $M'Q=MQ+L'Q\subseteq 0+$

$L'P = 0$, so $M' \subseteq K'$. Since

$$M'/L' = M + L'/L' \simeq M/M \cap L' = M/M \cap \ell_{E(M)}(P) = M/\ell_M(P) = M/L,$$

M'/L' is uniform, and since $M'/L' \subseteq K'/L' \simeq \bigoplus_{i=1}^n K_i/L_i$, M'/L' embeds in K_i/L_i for some i, say $M'/L' \simeq M_i/L_i \subseteq K_i/L_i$. Now M_i is a P-tame uniform right R-module with an affiliated series $0 \subset L_i \subset M_i$, affiliated primes and P and Q such that M_i/L_i is uniform, $Q \subset P$, and $M_iQ = 0$. As at the beginning of this proof, it may be assumed that M_i is finitely generated, and this contradicts (ii). □

In [6, p. 188], Jategaonkar introduced a condition for a prime ideal P of a right noetherian ring R that was designed to rule out the occurrence of the undesirable case of Lemma 5.1 for a P-tame right R-module M. The condition postulates that the second layer of M, that is, the module $M/\ell_M(P)$ be tame, hence the name *second layer condition*.

Lemma 5.3. *The following statements are equivalent for a prime ideal P of a right noetherian ring R.*

(i) *The second layer of any P-tame right R-module is tame.*
(ii) *The second layer of any uniform P-tame right R-module is tame.*
(iii) *The second layer of E_P is tame.*
(iv) *The second layer of $E_R(R/P)$ is tame.*

Proof. The implication (i) ⇒ (ii) is trivial, and so is (ii) ⇒ (iii), since E_P is uniform and P-tame.

(iii) ⇒ (iv): Since $E_R(R/P) \simeq E_P^n$ for some integer $n > 0$, it follows that $\ell_{E_R(R/P)}(P) \simeq (\ell_{E_P}(P))^n$, so $E_R(R/P)/\ell_{E_R(R/P)}(P) \simeq (E_P/\ell_{E_P}(P))^n$, whence the claim, since direct sums of tame modules are tame.

(iv) ⇒ (i): Since a P-tame right R-module M embeds in a direct sum of copies of $E_R(R/P)$, the second layer of M embeds in a direct sum of copies of the second layer of $E_R(R/P)$. □

Definition. A prime ideal P of a right noetherian ring R satisfies the *right second layer condition* if the second layer of $E_R(R/P)$ is tame. A set $\mathscr{X}$ of prime ideals satisfies the *right second layer condition* if every $P \in \mathscr{X}$ satisfies this condition. The ring R satisfies the *right second layer condition* if Spec(R) satisfies it.

Proposition 5.4. *Consider the following statements for a prime ideal P of a right noetherian ring R.*

(sslc) *P satisfies the right strong second layer condition.*
(rsslc) *P satisfies the right restricted strong second layer condition.*
(slc) *P satisfies the right second layer condition.*
(aslc) *P satisfies the right affiliated second layer condition.*
(saslc) *P satisfies the right strong affiliated second layer condition.*

Then

(i) (sslc) $\Rightarrow$ (rsslc) $\Rightarrow$ (slc) $\Rightarrow$ (aslc).

(ii) (aslc) $\Rightarrow$ (slc), *if* R *is noetherian.*

(iii) (slc) $\not\Rightarrow$ (rsslc) *in general, even when* R *is noetherian.*

(iv) (sslc) $\Rightarrow$ (saslc) $\Rightarrow$ (aslc).

Proof. (i) The first implication follows from the definition of the strong second layer condition and that of its version restricted to tame modules. The second implication was first observed by Kosler [7, Lemma 2.5]. Assume the right restricted strong second layer condition for P, set $L = \ell_{E_P}(P)$, let $\overline{M} = M + L/L$ be a finitely generated submodule of E_P/L, and let $\overline{S}$ be a subset of $\overline{M}$. Then $S = \{s \mid s + L \in \overline{S}\}$ is a subset of the finitely generated P-tame right R-module M, which is a Δ-module by Theorem 4.2. Consequently, $r(S) = \bigcap_{i=1}^{n} r(s_i)$, $s_i \in S$. Let $r \in \bigcap_{i=1}^{n} r(s_i + L)$. Then $s_i rP = 0$ for all i, so $SrP = 0$, so $Sr \subseteq L$, hence $r \in r(\overline{S})$. This proves that $\bigcap_{i=1}^{n} r(s_i + L) \subseteq r(\overline{S})$, and since the reverse inclusion is trivial, $\overline{M}$ is therefore a Δ-module, hence tame. Consequently, the second layer of E_P is tame.

For the last implication, assume that there exists a finitely generated P-tame uniform right R-module M with an affiliated series $0 \subset L \subset M$ and corresponding affiliated prime ideals P and Q such that M/L is uniform, $Q \subset P$, and $MQ = 0$. By Lemma 5.1, this is an incidence of the undesirable case, so M/L is wild. However, $\mathrm{E}(M) \simeq E_P$, so

$$M/L = M/\ell_M(P) = M/M \cap \ell_{E_P}(P) \simeq M + \ell_{E_P}(P)/\ell_{E_P}(P) \subseteq E_P/\ell_{E_P}(P).$$

Since the second layer of E_P is assumed to be tame, this gives a contradiction.

(ii) Assume that P satisfies the right affiliated second layer condition. Set $L = \ell_{E_P}(P)$, and let U/L be a uniform submodule of E_P/L. Among the submodules of U that are not contained in L, choose one with maximal right annihilator, say M. Then M satisfies the hypotheses of Lemma 5.1. The hypothesis for P means that the desirable case occurs, and since R is assumed to be noetherian, this means that M/L is tame. As $M/L \subseteq_{\mathrm{ess}} U/L$, U/L is also tame. Thus, the second layer of E_P is tame.

(iii) It has already been noted above that the prime ideal P of the noetherian ring T in [3, Exercise 11M] satisfies the right strong affiliated second layer condition, hence also the right affiliated second layer condition, and hence the right second layer condition by (ii). But P does not satisfy the right restricted strong second layer condition.

(iv) Apply Proposition 3.1(v), Proposition 3.3 and Lemma 5.2. □

It is still an open question if the right affiliated second layer condition for a prime ideal P implies the right second layer condition for P when the ring R is merely right noetherian, in fact, it is not known if in this case the ring can satisfy the right affiliated second layer condition and fail to satisfy the right second layer condition. In contrast, it is shown below that while a prime ideal of a noetherian ring may satisfy the right second layer condition but fail to satisfy the right restricted strong second layer condition, a right link closed set of prime ideals of a right noetherian ring R that satisfies the right second layer condition also satisfies the right restricted strong second layer condition. Since Spec(R) is trivially right link closed, the two conditions are thus equivalent when imposed on the ring as a whole.

Definition. Let $\mathscr{X}$ be a set of prime ideals, and let M be a right R-module. A sequence of submodules

$$0 = N_0 \subset N_1 \subset \cdots \subset N_i \subset N_{i+1} \subset \cdots \subset N_k = M$$

is called an *$\mathscr{X}$-tame prime factor series* of M if each of the factors N_i/N_{i-1} is tame and P_i-prime for some $P_i \in \mathscr{X}$. If $\mathscr{X} = \text{Spec}(R)$, then such a series is simply called a *tame prime factor series* of M.

Proposition 5.5. *The following are equivalent for a right link closed set $\mathscr{X}$ of prime ideals of the right noetherian ring R.*

(i) *$\mathscr{X}$ satisfies the right restricted strong second layer condition.*

(ii) *$\mathscr{X}$ satisfies the right second layer condition.*

(iii) *Every finitely generated $\mathscr{X}$-tame right R-module has an $\mathscr{X}$-tame prime factor series.*

Proof. It follows from Proposition 5.4 (i) that (i) implies (ii).

(ii)$\Rightarrow$(iii): This is a slightly stronger version of [6, Lemma 7.1.2]. See also [7, Proposition 2.4]. Let M be a finitely generated $\mathscr{X}$-tame right R-module, and assume that M embeds in a module N that has an $\mathscr{X}$-tame prime factor series $\{N_i\}$. It is easy to see that the distinct terms of the sequence $\{M \cap N_i\}$ form an $\mathscr{X}$-tame prime factor series of M. Now, proceed by noetherian induction, assuming that all $\mathscr{X}$-tame proper homomorphic images of M have an $\mathscr{X}$-tame prime factor series. If M is not uniform, then M embeds in a finite direct sum of $\mathscr{X}$-tame proper homomorphic images M/N_i of M. Each of these has an $\mathscr{X}$-tame prime factor series by the inductive hypothesis, so their direct sum has such a series, so M has one by the remark above. Thus, let M be uniform, let $\text{Ass}(M) = P \in \mathscr{X}$, set $L = \ell_M(P)$, and observe that M/L embeds in the second layer of $\text{E}_R(R/P)$. Since P satisfies the right second layer condition, M/L is thus tame, and Jategaonkar's Main Lemma [6, 6.1.3] shows that M/L is in fact $\mathscr{X}$-tame. By induction, M/L has an $\mathscr{X}$-tame prime factor series, which together with L yields such a series for M.

(iii)$\Rightarrow$(i): (See [7, Proposition 2.6]) Let $P \in \mathscr{X}$, and let $M \subseteq E_P$ be finitely generated with prime annihilator $r(M) = Q \subseteq P$. We have to show that $Q = P$. Let $0 = M_0 \subset \cdots M_i \subset \cdots \subset M_k = M$, be an $\mathscr{X}$-tame prime factor series of M with $P_i = r(M_i/M_{i-1}) = \text{Ass}(M_i/M_{i-1})$. Since $P_i \supseteq Q$ for all i, and since $P_k P_{k-1} \cdots P_2 P_1 \subseteq r(M) = Q$, $P_i = Q$ for some i. Then $Q = r(M) \subseteq r(M_i) \subseteq r(M_i/M_{i-1}) = P_i = Q$, whence $r(M_i) = P_i = Q$. Assume that $i > 1$. Since M is uniform, $M_{i-1} \subseteq_{\text{ess}} M_i$ as R/P_i-modules, which contradicts the fact that M_i/M_{i-1} is $\mathscr{C}(P_i)$-torsionfree. Thus, $Q = P_1 = P$. $\square$

Since Spec(R) is trivially right link closed, we have the following.

Corollary 5.6. *A right noetherian ring satisfies the right restricted strong second layer condition if and only if it satisfies the right second layer condition.*

6. Relative fully boundedness

Right fully bounded right noetherian rings have been described in many ways. The most important characterization has been achieved by Cauchon [2, Théorème III 6], who proved that a right noetherian ring R is right fully bounded if and only if every finitely generated right R-module is a Δ-module. Perhaps less well-known is the following characterization, which can easily be derived from [9, Satz 3.2]: a right noetherian ring R is right fully bounded if and only if $|M| = \text{Cl.K.dim}(R/r(M))$ for every finitely generated right R-module M. Since, as a consequence of Theorem 4.2 and Proposition 5.5, a right noetherian ring R satisfies the right second layer condition iff it is right fully tame bounded iff every finitely generated tame right R-module is a Δ-module, it is natural to ask whether a characterization similar to the one above can also be obtained. This is indeed possible, as we proceed to show. However, a dimension other than the Krull dimension has to be used, since, for example, the first Weyl algebra A_1 satisfies the right second layer condition, is tame as a right module over itself, yet $|A_1| = 1 > 0 = \text{Cl.K.dim}(A_1) = \text{Cl.K.dim}(A_1/r(A_1))$. Now, [8] Kosler, introduced the following concept for a module M that he named the *classical Krull dimension* of M denoted by $\text{Cldim}(M)$. We prefer the name *tame dimension* and will denote it by $\gamma(M)$, since the term classical Krull dimension suggests a very close relationship in general with the classical Krull dimension defined for rings. While such a relationship does indeed exist, it seems to be restricted to rings with the second layer condition.

Definition. Let R be a right noetherian ring. For a right R-module M set

$$\Gamma(M) = \{T \mid T \subseteq M,\ M/T \text{ is tame }\}.$$

The ordinal $\gamma(M) = \text{dev}(\Gamma(M))$ is called the *tame dimension* of M. Thus $\gamma(M) = -1$ if no homomorphic image of M is tame and $\gamma(M) = \alpha$ for an ordinal $\alpha > -1$ if $\gamma(M) \not< \alpha$ and, given any infinite descending chain $M = T_0 \supset T_1 \supset \cdots \supset T_i \supset T_{i+1} \supset \cdots$ with $T_i \in \Gamma(M)$, then $\gamma(T_i/T_{i+1}) < \alpha$ for all but finitely many indices i.

It is obvious that $\gamma(M)$ is defined for any module M with Krull dimension, and that $\gamma(M) \leq |M|$. Note that if R is a right noetherian right fully bounded ring then γ is just the Krull dimension, since in this case every right R-module is tame. In general, one has that $\max\{\gamma(N), \gamma(M/N)\} \leq \gamma(M)$ for a submodule $N \subseteq M$ [8, Proposition 2.3]. Whether the tame dimension is exact, that is, whether the above inequality can be sharpened to an equality in general, is an open question. Kosler [8, Theorem 2.4] establishes this, provided R satisfies the right second layer condition, and the proof is surprisingly difficult. The name *classical Krull dimension* for γ was motivated by the fact [8, Theorem 2.9] that $\gamma(R_R) = \text{Cl.K.dim}(R)$ for a right noetherian ring R that satisfies the right second layer condition.

The following is a sharpening of [8, Proposition 2.7].

Lemma 6.1. *Let E be an essential right ideal of the right noetherian prime ring R. If R/E is tame then $\gamma(R/E) < \gamma(R_R)$.*

Proof. Set $E_0 = R$, $E_1 = E$, and assume that for $k \geq 1$ essential right ideals $E_i, 1 \leq i \leq k$ have been found such that $E_{i-1} \supset E_i$, R/E_i is tame and $\gamma(E_{i-1}/E_i) \geq \beta = \gamma(R/E)$. Let $c \in E_k$ be a regular element, and choose a submodule $E_{k+1}/cE_k \subseteq E_k/cE_k$ that is maximal with respect to $(cR/cE_k) \oplus (E_{k+1}/cE_k) \subset_{\text{ess}} E_k/cE_k$. Then

$$R/E_k \simeq cR/cE_k \simeq \frac{cR + E_{k+1}}{E_{k+1}} \subseteq_{\text{ess}} E_k/E_{k+1}.$$

Since R/E_k is tame, its essential extension E_k/E_{k+1} is tame. Since extensions of tame modules by tame modules are tame, R/E_{k+1} is tame. Note that $\gamma(E_k/E_{k+1}) \geq \gamma(R/E_k) \geq \gamma(E_{k-1}/E_k) \geq \beta$. Since $c^2R \subseteq cE_k \subseteq E_{k+1}$, E_{k+1} is an essential right ideal. Thus, there exists an infinite descending chain $R \supset E_1 \supset \cdots \supset E_i \supset E_{i+1} \supset \cdots$, such that each R/E_i is tame and $\gamma(E_i/E_{i+1}) \geq \beta$ for all i. Consequently, $\gamma(R_R) > \beta = \gamma(R/E)$. □

The foregoing now allows to characterize the right second layer condition for a right noetherian ring R by the property that $\gamma(M) = \text{Cl.K.dim}(R/r(M))$ for any finitely generated tame right R-module M. Most of the statements of the following result have already been proved above. In view of Theorem 4.2 and Proposition 5.5, the list could even be longer, but we have chosen to include only those statements that emphasize the formal similarity between fully boundedness and the second layer condition.

Theorem 6.2. *The following statements are equivalent for a right noetherian ring R.*

(i) *R satisfies the right second layer condition.*

(ii) *Every finitely generated tame right R-module is a Δ-module.*

(iii) *$|M| = |R/r(M)|_R$ for every finitely generated tame right R-module M.*

(iv) *R is right fully tame bounded.*

(v) *$\gamma(M) = \text{Cl.K.dim}(R/r(M))$ for every finitely generated tame right R-module M.*

Proof. The equivalence of statements (i)–(iv) follows from Theorem 4.2 and Proposition 5.5. For (iii), note that $|M| = \kappa_\tau(M)$, τ the torsion theory where all non-zero right R-modules are torsionfree, so that $\text{Spec}_\tau(R) = \text{Spec}(R)$.

(i) ⇒ (v): $\gamma(M) = \gamma(R/r(M))$, by [8, Corollary 2.5], and $\gamma(R/r(M)) = \text{Cl.K.dim}(R/r(M))$, by [8, Theorem 2.9],

(v) ⇒ (iv): Let P be a prime ideal, and let E/P be an essential right ideal of R/P, such that R/E is tame. Note that R/P is also tame. Applying (v) and Lemma 6.1 gives

$$\text{Cl.K.dim}(R/r(R/E)) = \gamma(R/E) < \gamma(R/P) = \text{Cl.K.dim}(R/P),$$

so that $r(R/E) \supset P$. □

Naturally, the question arises whether the right strong second layer condition can be characterized in a similar way. Obviously, the class of finitely generated tame right R-modules of the above characterizations would have to be replaced by a larger class of modules. This appears to be a difficult problem, particularly in view of the fact that to date no example is known of a noetherian ring that satisfies the second layer condition, but does not satisfy the strong second layer condition. So far, we only have the following partial result in this direction.

Proposition 6.3. *The following statements are equivalent for a right noetherian ring R.*

(i) *R satisfies the right strong second layer condition*

(ii) *Given any prime ideal P of R, every essential right ideal E/P of R/P satisfying $P \notin \mathrm{Ass}(R/E)$ contains a nonzero two-sided ideal of R/P.*

Proof. (i) $\Rightarrow$ (ii): Let P be a prime ideal, let E/P be an essential right ideal of R/P, and assume that $P \notin \mathrm{Ass}(R/E) = \{P_1, \dots, P_n\}$. There exist right ideals E_i, $i = 1, \dots, n$, such that $\mathrm{Ass}(R/E_i) = P_i$ and $E = \bigcap_{i=1}^n E_i$. Since each P_i is assumed to satisfy the right strong second layer condition, $r(R/E_i) \supset P$ for each i, whence $r(R/E) = \bigcap_{i=1}^n r(R/E_i) \supset P$.

(ii) $\Rightarrow$ (i): Let $Q \subset P$ be prime ideals, and let $M = \sum_{i=1}^n m_i R$ be a finitely generated right R/Q-module with $\mathrm{Ass}(M_R) = P$. We have to show that $r_R(M) = \bigcap_{i=1}^n r(m_i R) \supset Q$, so we may reduce to the case where M is cyclic, $M = mR$. Obviously, $Q \notin \mathrm{Ass}(R/r(m)) = P$, so $r(m)/Q \subseteq_{\mathrm{ess}} R/Q$, and the claim follows from (ii). □

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